Math 10B with Professor Stankova
Worksheet, Discussion #9; Thursday, 2/21/2019
GSI name: Roy Zhao

1 Conditional Probability

1.1 Concepts

1. The probability of an event A given that we know an event B occurred is denoted as $P(A|B)$ and the formula is
$$P(A|B) := \frac{P(A \cap B)}{P(B)}.$$
Sometimes, we do not know $P(B)$ and we have to rewrite it as
$$P(B) = P(B \cap A) + P(B \cap \bar{A}) = P(B|A)P(A) + P(B|\bar{A})P(\bar{A})$$

1.2 Examples

2. I flip a fair coin 12 times. What is the probability that exactly 10 heads appear given that at least two heads appeared?

Solution: We need to find the probability $P(10H| \geq 2H)$ and by definition, we need to find $P(10H \cap \geq 2H)$ and $P(\geq 2H)$. The event that we get at least 2 heads at exactly 10 heads is just 10 heads and we can calculate the probability that we have at least 2 heads by complementary counting. This probability is $1 - P(< 2H) = 1 - P(0H) - P(1H) = 1 - \frac{1}{2^{12}} - \frac{12}{2^{12}} = 1 - \frac{13}{2^{12}}$. Thus, we have that
$$P(10H| \geq 2H) = \frac{P(10H)}{P(\geq 2H)} = \frac{\frac{\binom{12}{10}}{2^{12}}}{1 - \frac{13}{2^{12}}} = \frac{\binom{12}{10}}{2^12 - 13}.$$

1.3 Problems

3. **TRUE** False If $A \subset B$, then $P(A|B) = P(A)/P(B)$.

Solution: If $A \subset B$, then $A \cap B = A$.

4. True **FALSE** If $P(A|B) = P(B|A)$, then $P(A) = P(B)$.

Solution: If $P(A \cap B) = 0$, then both conditional probabilities are 0 and we don't need $P(A) = P(B)$.

5. Suppose that 80% of students have taken Calculus and of those, only 20% become math graduate students. Suppose that 30% of all students become math graduate students. Let C be the set of students who have taken Calculus and let G be the set of students who become math graduate students. Represent $80\%, 20\%, 30\%$ in terms of probabilities (e.g. $P(A)$ or $P(A|B)$).

Solution: $P(C) = 80\%, P(G|C) = 20\%, P(G) = 30\%$.

6. Using the previous problem, what is the probability that someone has not taken calculus and is a math graduate student? What is the probability that a math graduate student has not taken calculus?

Solution: $P(\bar{C} \cap G) = P(G) - P(C \cap G) = P(G) - P(G|C)P(C) = 30\% - 20\%80\% = 30 - 16 = 14\%$. $P(\bar{C}|G) = \frac{P(\bar{C} \cap G)}{P(G)} = \frac{14}{30} = \frac{7}{15}$.

7. Out of 330 male students and 270 female students in 10B, 210 of the men and 180 of the women took 10A with Zvezda last semester. What is the probability that a randomly person is a female given that they took 10A with Zvezda?

Solution: We have that $P(F|Zvezda) = \frac{P(F \cap Zvezda)}{P(Zvezda)} = \frac{P(F \cap Zvezda)}{P(Zvezda \cap M) + P(Zvezda \cap F)} = \frac{180}{210+180} = \frac{18}{39} = \frac{6}{13}$.

8. There are two boxes. One has 10 red balls and no white balls and one has 5 red and 5 white balls. I randomly pick a box then randomly pick out a ball. What is the probability I pick out a red ball?

Solution:

$$\begin{aligned} P(\text{red}) &= P(\text{red} \mid \text{box } 1)P(\text{box } 1) + P(\text{red} \mid \text{box } 2)P(\text{box } 2) \\ &= 1 \cdot \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{4}. \end{aligned}$$

9. What is the probability that I have a full house (one triple, one pair) in my 5 card hand if I know that I have a triple (but not 4 of a kind)?

Solution:

$$P(Full|Triple) = \frac{P(Full \cap Triple)}{P(Triple)} = \frac{P(Full)}{P(Triple)}$$
$$= \frac{\frac{13\binom{4}{3}\cdot 12\binom{4}{2}}{\binom{52}{5}}}{\frac{13\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}}$$
$$= \frac{12\binom{4}{2}}{\binom{48}{2}} = \frac{3}{47}.$$

2 Review

10. How many ways can you rearrange the letters in BERKELEY?

Solution: There are 8 letters but 3 E's and no other letters repeat. So the answer is $\frac{8!}{3!}$.

11. There are 72 students trying to get into 3 of my sections. There are $27, 20, 25$ openings respectively. How many ways are there for these students to enroll?

Solution: $\binom{72}{27}\binom{72-27}{20}\binom{72-27-20}{25} = \binom{72}{27}\binom{45}{20}$.

12. How many ways can I put 20 Tootsie rolls into 5 goodie bags so that each goodie bag has at least 2 Tootsie roll?

Solution: Put one in each then there are 15 rolls left and 5 bags and everything is indistinguishable so $p_5(15)$.

13. Show that when you place 9 coins on an 8×10 boards, at least two coins must be on the same row.

Solution: PP, $9/8 > 1$.

14. How many different three-letter initials contain A?

Solution: We use complementary counting to get $26^3 - 25^3$.

15. How many license plates with 3 digits followed by 3 letters do not contain the both the number 0 and the letter O (it could have an O or a 0 but not both).

Solution: Complementary counting. Bad cases are if it has an O and a 0. There are $26^3 - 25^3$ ways to have a O, there are $10^3 - 9^3$ ways to have a 0. So the final answer is

$$10^3 \cdot 26^3 - (10^3 - 9^3)(26^3 - 25^3).$$

16. Prove that $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$ in two different ways.

Solution: We can first prove it algebraically by

$$\begin{aligned}\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!} \\ &= \frac{1}{n-r}\frac{(n-1)!}{(r-1)!(n-r-1)!} + \frac{1}{r}\frac{(n-1)!}{(r-1)!(n-r-1)!} \\ &= \frac{r+n-r}{r(n-r)}\frac{(n-1)!}{(r-1)!(n-r-1)!} \\ &= \frac{n!}{r!(n-r)!} = \binom{n}{r}.\end{aligned}$$

Then, to prove it combinatorially, we see that the right side is just choosing a team of r people out of n total people. We can do this by first distinguishing one person out, say Zvezda. Then, if she is on the team, out of the remaining $n-1$ people, we still need to choose $r-1$ which can be done in $\binom{n-1}{r-1}$ ways. Otherwise, if she is not on the team, we still need to choose r people which can be done in $\binom{n-1}{r}$ ways. Since both count the number of ways, the left and right side are equal.

17. Prove that $\sum_{k=0}^{n} 5^k \binom{n}{k} = 5^0\binom{n}{0} + 5^1\binom{n}{1} + \cdots + 5^n\binom{n}{n} = 6^n$.

Solution: Use the Binomial theorem and plug in $x = 1, y = 5$.

18. How many ways can I split up 30 distinguishable students into 6 groups each of size 5?

Solution: $\frac{30!}{5!5!5!5!5!5!}$.

19. Find a formula for $1 + 2 + 4 + \cdots + 2^n$ and prove it.

Solution: The answer is $2^{n+1} - 1$ and use induction to prove it.